# A Game Theory Approach to Evaluating Transport Network Connectivity and Reliability

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### Abstract

In this paper, we will introduce some practical problems on transport networks, where there are some participants who want to travel from an origin node to a destination node. To measure network connectivity and vulnerability under unexpected incidents like nature disasters, we estabilish a two-player zero-sum game model, and evaluate the network reliability by the expected travel cost at the equilibrium of this game. Also, we introduced one-traveller model, multiple-traveller model and relaxed pessimism model in this game for different real-world modelings.


## Introduction

As a mathematical background for the analysis of interactive processes for decision-making, game theory is more and more widely apllied in the field of transport network analysis. On a transport network, one or more travellers wish to travel from their origin node to destination node with lowest cost, such as distance, time, etc. Recent researches mainly focus on two topics:

One is the traffic user equilibrium, which determines the traffic flow on every link of the network; Another main application topic is reliability problems, which focuses on the network performance and response under the disability of one or more links. This is a trending topic nowadays, and many researchers study network reliability, or vulnerability with different models, settings and measures.

A most conservative way is to calculate the expected time cost of each path according to the probability of failure; however, in most real situations failure of links is rare, so it's hard to estimate the probability of each link.

To avoid the most extreme cases, one paradigm views the problem as a game, where travellers or network dispatcher determine paths for traffic flow in the expectation to minimize the expected cost, and another agent viewed as the nature or an evil attacker, who tries to maximize the cost by damaging one or more links, is introduced. So this becomes a two-player zero-sum game, and we can evaluate the network reliability by the expected travel time at the equilibrium of this game.



In the next, we will discuss the main methods and results from different researches. They measures the network vulnerability under various transport problems. The first model only considers the behavior of one traveller, and the cases of multiple travellers are discussed later.

## Vulnerability from Individual Perspective

In the earliest study of this field (Bell 1999), the author proposed a model containing two players in one game: a traveller that's going to travel from an origin node to a destination node in a graph, and the nature as an opponent that causes an incident on one or several links in the graph. Both agents in this game adopt mixed strategies, in the expectation to minimize (the traveller) or maximize (the nature) the expected travel time. This turns out to be a two-player zero-sum game, and a Nash equilibrium in this game is promised to exist.

Formally speaking, we have links corresponding to edges of the graph, and paths from origin to destination. The information about all possible paths is contained in the incidence matrix: $a_{ik} = 1$ if link $i$ is on path $k$; 0 otherwise. Also, we need the edge-vertex incidence matrix: $e_{ni} = 1$ if link $i$ enters node $n$, $-1$ if leaves node $n$, and 0 otherwise. So a valid path $k$ should satsify $\sum_i a_{ik}e_{ni} = b_n$, where $b_n = 1$ if node $n$ is the destination; $-1$ if $n$ is the origin, and 0 otherwise.

(Note: in the original paper $b_n = 1$ if $n$ is origin or destination, which is an incorrect definition.)

And the opponent's choices are expressed as differert scenarios, and the link $i$ has link cost $c_{ij}$ under the scenario $j$. With this setting, the nature's choice of destorying links can be transferred into choices of different scenarios in which the influenced links will lead to high link cost.

Above all, we have established the traveller's srategies as choosing different paths, and the nature's strategies as choosing different scenarios. Both agents can adopt mixed strategies. Denote $h_k$ as the probability that path $k$ is taken, $q_j$ as the probability of scenario $j$. Also, we define $p_i$ as the probability that link $i$ is taken, then we have the following equations:

$$p_i = \sum_k h_k a_{ik}, \forall i \tag{1}$$

Also, we define $g_{kj}$ of the cost for path $k$ under scenario $j$, and $TC$ as the expected total cost. So we have:

$$g_{kj} = \sum_i a_{ik} c_{ij} \quad (2)$$

$$TC = \sum_i \sum_j p_i q_j c_{ij} = \sum_k \sum_j h_k q_j g_{kj} \quad (3)$$

So we can naturally draw a solution to this problem: first enumerate all paths and derive the $a_{ik}$ matrix and thus $g_{kj}$ matrix as (2). Then from the minmax theorem, we can find a equilibrium point of (3).

However, when the graph becomes large, enumerating all paths and generate corresponding incidence matrices in the graph can be complex and costly. Hence, the author proposed an iterating solution, which is named as the method of successive averages(MSA), to solve the problem efficiently. The idea is to play according to the opponent's history moves. Suppose the strategies in the $k$-th round are $\mathbf{h}^{(k)}$ and $\mathbf{q}^{(k)}$ respectively. If this is not an equilibrium, either the traveller and the nature can find a pure stratetgy best response. Formally writing, the iteration algorithm is:

1. Given the nature's strategy $\mathbf{q}^{(k)}$, the traveller chooses his best response as $\mathbf{x}^{(k)}$. We may only consider pure strategies here, so here we can calculate the expected link cost of each link, and apply shortest path algorithm minimizing the expected cost instead of enumerating all possible paths.
2. The traveller adopts $\mathbf{h}^{(k+1)} = \frac{k}{k+1}\mathbf{h}^{(k)} + \frac{1}{k+1}\mathbf{x}^{(k)}$.
3. The nature finds the best response to $\mathbf{h}^{(k+1)}$, as $\mathbf{y}^{(k)}$.
4. The nature adopts $\mathbf{q}^{(k+1)} = \frac{k}{k+1}\mathbf{q}^{(k)} + \frac{1}{k+1}\mathbf{y}^{(k)}$.
5. $k \leftarrow k+1$, and we start another round of iteration until convergence.

In numerical examples, the author found that the value of expected time converges fast, while the strategies for the two players converge slowly. This may refelct the non-uniqueness of equilibrium strategies.

In his later work (Bell 2003), the author commented that although he hadn't found a proof of convergence, an example where convergence to an equilibrium trip cost does not occur rapidly hadn't been discovered.

## Reliability from the System Perspective

In the above paper the author made an assumption that the link cost is traffic-independent, so that we only need to look into one participant. This seems true when the measure is travelling distance or something else, but in most cases there's no reason to make the assumption that different Origin-Destination pairs may be considered separately since the travel cost, for example the travel time can be traffic-dependent. When the traffic is heavy on some links, the time cost to pass the link can be longer, which is consistent with our real-life experience.

Another paper *A game theory approach for the measurement of transport network vulnerability from the system prospective* (Qiao et al. 2014) answers the question when there're multiple agents in one network. They mainly focused on the total travel time as a system-level measure, and they adopted a more practical model of a BPR (Bureau of Public Roads) function between link time cost and link traffic flow:

$$t(x) = f(1 + \alpha(x/c)^\beta) \quad (4)$$

where $f$ measures the free flow time through this link, $c$ measures the capacity of the link. These two parameters can vary for different links and scenarions. There're also fixed parameters $\alpha, \beta$, and $x$ is the traffic flow on this link.

In this model, there're multiple sources and sinks with various flow amounts $d_n$ (positive for sink nodes, negative for source nodes) in this network, and we're going to dispatch the flows under the threaten of different failure scenarios, so as to reach the minimum expected cost. This setting seems to be not that practical since in most real cases the O-D pairs are designated for travellers, but it's simple to analyze since there's only one objective function in the optimization, and it suffices to measure the network vulnerability.

Set the traffic flow on the link $i$ as $x_i$, the probability of scenario $j$ as $y_j$, and the capacity parameter of link $i$ under scenario $j$ as $c_{ij}$. So the minmax problem can be formulated as:

$$\min_{x_i} \max_{y_j} x_i t_{ij}(x_i) y_j \quad (5)$$

s.t.

$$\sum_j y_j = 1, \sum_i e_{ni} x_i = d_n, \forall n, x_i \geq 0, y_j \geq 0. \quad (6)$$

The minmax problem can be broken into the following bi-level optimization problem. In the upper level, given a fixed link flow strategy, the opponent tries to maximize the expected cost by adjusting the disturbing probabilities; In the lower level, the network dispatcher performs a system optimal assignment given the link disturbance probabilities.

Upper level:

$$\max_{y_j} \sum_i \sum_j x_i t_{ij}(x_i) y_j, \quad (7)$$

s.t.

$$\sum_j y_j = 1, y_j \geq 0. \quad (8)$$

Lower level:

$$\min_{x_i} \sum_i \sum_j x_i t_{ij}(x_i) y_j, \quad (9)$$

s.t.

$$\sum_i e_{ni} x_i = d_n, \forall n, x_i \geq 0 \quad (10)$$

The upper level is simply a linear programming problem, whose solution is promised to exist. The existence of the lower level optimization is not that obvious; in fact, the constraints are convex, and we can prove that the objective function is also convex:

We have the entries in the Hessian matrix

$$H_{m,n} = \mathbf{1}_{m=n} \frac{\partial^2}{\partial x_m \partial x_n} \left( \sum_i \sum_j x_i t_{ij}(x_i) y_j \right) \quad (11)$$

$$= \mathbf{1}_{m=n} \sum_j (2t'_{mj}(x_m) + x_m t''_{mj}(x_m)) y_j. \quad (12)$$

So the determinant of the Hessian matrix is

$$\det H = \prod_m \left( \sum_j (2t'_{mj}(x_m) + x_m t''_{mj}(x_m)) y_j \right). \quad (13)$$

And from the BPR cost function,

$$2t'_{mj}(x_m) + x_m t''_{mj}(x_m) = \frac{f_{mj} \alpha \beta (\beta + 1)}{c_{mj}^{\beta}} x_{mj}^{\beta - 1} \geq 0. \quad (14)$$

Thus $\det H \geq 0$, implying the convexity of the objective function. From the convex optimization techniques, we know that we're able to find the unique system optimal assignment with, for example gradient descent method.

In order to solve the equilibrium problem, the author proposed an iteration algorithm which is similar to the MSA method discussed in the last section:

1. Initialize $\mathbf{y}^{(0)}$ as choosing all scenarios with equal probability. Solve $\mathbf{x}^{(0)}$ as the system optimal assignment under $\mathbf{y}^{(0)}$.
2. Given the assignment $\mathbf{x}^{(k)}$, the nature chooses the most critical path to disturb, that is, to find
$$j = \arg\max_j \sum_i x_i^{(k)} t_{ij}(x_i^{(k)}).$$
denote the pure strategy of scenario $j$ as $\mathbf{y}'^{(k+1)}$, and update $\mathbf{y}^{(k+1)}$ with MSA:
$$\mathbf{y}^{(k+1)} = \frac{k}{k+1} \mathbf{y}^{(k)} + \frac{1}{k+1} \mathbf{y}'^{(k+1)}.$$
3. Given the new scenario probabilities $\mathbf{y}^{(k+1)}$ in the above, the dispatcher looks for a new system optimal assignment $\mathbf{x}'^{(k+1)}$ and update $\mathbf{x}^{(k+1)}$ using MSA:
$$\mathbf{x}^{(k+1)} = \frac{k}{k+1} \mathbf{x}^{(k)} + \frac{1}{k+1} \mathbf{x}'^{(k+1)}.$$
4. Go to step 2 if the stopping criteria are not satisfied.

Convergence of this iterating algorithm is still yet to be proved.

Using this method, the author analyzed an example and pointed out that two parallel roads may work better than a single link with doubled capacity, even if the failure penalty (capacity reduction) are the same in the two cases. This may be heuristic to our real-life road designing.

## Let's get relaxed: Measuring Our Pessimism about the Network

It's necessary to point out that, in the above, the travellers or the dispatcher always aims at improving the performance at the worst case, which shows our extreme pessimism. But sometimes we can relax our the degree of pessimism by adding a weighted entropy function, as Bell discussed in his later work (Bell 2014).

Take the first one-traveller model as an example: The author suggests to add an entropy term to the cost function, as:

$$TC = \sum_k \sum_j h_k q_j g_{kj} - \frac{1}{\theta} \sum_j q_j \ln q_j, \quad (15)$$

here $\theta$ measures out pessimism in the way that the larger $\theta$ is, the more pessimistic we're.

In this case, the explicit solution for $q_i$ is easy to find:

$$q_j = \frac{\exp(\theta \sum_k h_k g_{jk})}{\sum_s \exp(\theta \sum_k h_k g_{jk})} \quad (16)$$

And thus

$$\min_{\mathbf{h}} TC(\mathbf{h}) = \min_{\mathbf{h}} \frac{1}{\theta} \ln \sum_j \exp\left(\theta \sum_k h_k g_{jk}\right) \quad (17)$$

s.t.

$$\sum_k h_k = 1, h_k \geq 0. \quad (18)$$

Note that the gradient of the objective function

$$\frac{\partial}{\partial h_k} TC(\mathbf{h}) = \sum_j q_j g_{jk} \quad (19)$$

is exactly the expected travel cost via path $k$. So we can still find the path with lowest expectation cost and descent in this direction since the gradient is the steepest. This leads us to the following revised and shortened MSA:

1. Initialize $\mathbf{q}^{(0)}$ as choosing all scenarios with equal probability. $k \leftarrow 0$.
2. For a given scenario probability distribution $\mathbf{q}^{(k)}$, calculate the expected cost via each link and find the path with shortest cost using shortest path algorithms. Denote this auxiliary strategy as $\mathbf{y}^{(k+1)}$, and update the strategy using MSA:
$$\mathbf{h}^{(k+1)} = \frac{k}{k+1} \mathbf{h}^{(k)} + \frac{1}{k+1} \mathbf{y}^{(k+1)}$$
3. Calculate $\mathbf{q}^{(k+1)}$ given $\mathbf{h}^{(k+1)}$ via equation (16).
4. In the stop criteria are not satisfied, take $k \leftarrow k+1$ and start a new round of loop.

The author notified that the solution is unique in $\mathbf{q}$ since (15) is convex in $\mathbf{q}$, so we can find out the critical links and protect them accordingly.

## Conclusion

Above all, we have applied the two-player zero-sum game to analyze the vulnerability of a road network, and we're able to detect the sensitive links and strengthen the protection over them. However, we only discussed the model of one traveller and multiple travellers with undesignated origin-destination pairs, and the case of multiple travellers with designated O-D pairs is too complicated to be considered, since this problem is on the basis of multiple traveller equilibrium problem (or User Equilibrium, UE). Also, we only gave the efficient MSA algorithms under the three cases, which are proved useful in real-world examples. Although a counter-example hasn't been found, the convergence of the algorithms is still yet to be proved.

## References

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